Deformations of Isolated $\mathbb{Q}$-Factorial Toric Singularities

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Introduction

In deformation theory, the following question arises naturally: Given a scheme X with only *mild* singularities, what can we say about the singularities of local deformations of X? For X containing at worst isolated $\mathbb{Q}$-factorial toric singularities, any deformation of X over a complete DVR contains at worst isolated $\mathbb{Q}$-factorial toric singularities, independently of the characteristic. Over $\mathbb{C}$, this was conjectured by Riemenschneider and proven by Kollár and Shepherd-Barron (using the language of cyclic quotient singularities). Liedtke, Martin and Matsumoto established the connection to toric singularities and proposed a generalization to positive and mixed characteristic, that was recently proven by myself.

Main Theorem

Theorem. *Let X contain at worst isolated $\mathbb{Q}$-factorial toric singularities, and let $\mathcal{X} \to B$ be a local deformation of X as in the setup. Then the geometric generic fiber of $\mathcal{X} \to B$ contains at worst isolated $\mathbb{Q}$-factorial toric singularities.*

Proof - Main Ideas

0. In dimension ≥ 3 the singularities are rigid by [Sch71] (char 0) and [LMM25] (char $p > 0$) and the theorem therefore holds. We only need to look at toric surface singularities.
1. Reduce to singularity-deformations

$$\begin{array}{ccc} \operatorname{Spec} k[\![x_1, \dots, x_e]\!]/I & \longrightarrow & \operatorname{Spec} S[\![x_1, \dots, x_e]\!]/I' \\ \downarrow & \lrcorner & \downarrow \pi \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} S, \end{array}$$

where π is flat and surjective.

2. Use *shifting* automorphisms to find simple generators of I'.
3. Calculate the resulting singularities with the help of a new characterization of toric surface singularities (cf. lemma 1).

Toric Surface Singularities

The main theorem was already known in dimension ≥ 3, so we take a closer look at toric surface singularities. Here, the following two facts tell us that all toric surface singularities are isolated and $\mathbb{Q}$-factorial.

FACTS:

- The singular locus of toric varieties is always of codimension at least 2, so that toric surfaces can only have isolated singularities.
- A toric singularity is $\mathbb{Q}$-factorial iff the corresponding cone is generated by linearly independent vectors. This is always the case in dimension 2.

While toric surface singularities are in general not complete intersections, the following lemma tells us that in some sense, they are not far from it.

Lemma 1. *Let $I \subseteq k[\![x_1, \dots, x_e]\!]$ be a prime ideal of height $e-2$. If there exist $a_2, \dots, a_{e-1} \in \mathbb{N}$, such that $x_{i-1}x_{i+1} - x_i^{a_i} \in I$ for all $i = 2, \dots, e-1$, then $\operatorname{Spec} k[\![x_1, \dots, x_e]\!]/I$ is either smooth or a 2-dimensional toric surface singularity.*

This allows us to mostly concentrate on the $e-2$ generators of the form

$$x_1x_3 - x_2^{a_2}, \quad x_2x_4 - x_3^{a_3}, \quad \dots, \quad x_{e-2}x_e - x_{e-1}^{a_{e-1}}.$$

History of Relevant Results

The following table gives an overview of the progress made towards the theorem. The last three columns show the dimensions d, the embedding dimension e, and the characteristic of the underlying ground field $\operatorname{char} k$ for which the theorem was proven.

Authors		d	e	$\operatorname{char} k$
Schlessinger	[Sch71]	≥ 3	any	0
Riemenschneider	[Rie74]	2	≤ 5	0
Kollár and Shepherd-Barron	[KSB88]	2	any	0
Liedtke, Martin, and Matsumoto	[LMM25]	≥ 3	any	> 0
Pfeifer	[Pfe25]	2	any	any

Isolated $\mathbb{Q}$-factorial toric surface singularities are exactly the isolated cyclic (linearly reductive) quotient singularities. In fact, all papers cited here are written from the viewpoint of quotient singularities.

Setup

Let X proper over k contain at worst isolated $\mathbb{Q}$-factorial toric singularities, i.e. the non-smooth locus of X (if non-empty) is a finite set of $\mathbb{Q}$-factorial toric singularities. Let B be the spectrum of a complete DVR S and let $\mathcal{X} \to B$ be a flat morphism, such that

$$\begin{array}{ccc} X \cong \operatorname{Spec} k \times_B \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & \lrcorner & \downarrow \\ \operatorname{Spec} k & \longrightarrow & B, \end{array}$$

Assume furthermore that the non-smooth locus of $\mathcal{X} \to B$ is proper over B. We are interested in the singularities that can appear in the geometric generic fiber $\mathcal{X} \times_B \operatorname{Spec} \overline{\operatorname{Quot}(S)}$.

Shifting

Consider a singularity-deformation $\operatorname{Spec} S[\![x_1, \dots, x_e]\!]/I'$ whose defining ideal contains

$$x_1x_3 - x_2^3 + tx_1x_2, \qquad x_2x_4 - x_3^2, \qquad x_3x_5 - x_4^2.$$

The automorphism

$$x_3 \mapsto x_3 - tx_2$$

then transforms the elements above into

$$x_1x_3 - x_2^3, \qquad x_2x_4 - x_3^2 + 2tx_2x_3 - t^2x_2^2, \qquad x_3x_5 - x_4^2 - tx_2x_5.$$

This "shifts" the terms coming from the deformation "to the right".

Similar automorphisms can be found for general singularity-deformations of toric surface singularities and repeatedly applying them simplifies the terms generating I', giving us the following lemma.

Lemma 2. *After suitable automorphism, I' contains elements of the form*

$$x_{i-1}(x_{i+1} + tc_{i+1}) - x_i^{a_i} + tx_ih_i$$

for some $c_{i+1} \in S$ and $h_i \in S[x_i]$, $\deg_{x_i} h_i < a_i$.

Example

Let

$$X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]/(x_1x_3 - x_2^3, x_2x_4 - x_3^4, x_1x_4 - x_2^2x_3^3)$$

be the affine toric variety associated to the cone generated by $(0,1)^T$ and $(11,-7)^T$. Then a possible singularity-deformation is defined by the ideal

$$I' = (\, x_1x_3 - x_2^3 + tx_2^2,\ x_2x_4 - x_3^4 + tx_2,\ \dots\,).$$

While I' has 3 generators, we only need to inspect the first two.
Shifting gives us the automorphism $x_4 \mapsto x_4 - t$, which leads to

$$\begin{aligned} I' &= (\, x_1x_3 - x_2^3 + tx_2^2,\ x_2x_4 - x_3^4,\ \dots\,) \\ &= (\, x_1x_3 - x_2^2(x_2 + t),\ x_2x_4 - x_3^4,\ \dots\,). \end{aligned}$$

After passing to the geometric generic fiber, this defines an affine variety with two isolated toric singularities. Looking at the singularity at the origin, meaning that $(x_2 + t)$ becomes invertible, it is defined by the ideal

$$(\, x_1x_3 - x_2^2,\ x_2x_4 - x_3^4,\ \dots\,)$$

and is therefore isomorphic to the toric singularity coming from the cone with generators $(0,1)^T$ and $(7, -3^T)$.

References

[KSB88] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Inventiones mathematicae*, 91(2):299–338, Jun 1988.

[LMM25] Christian Liedtke, Gebhard Martin, and Yuya Matsumoto. Isolated quotient singularities in positive characteristic. *Astérisque*, 461, 2025.

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[Rie74] Oswald Riemenschneider. Deformationen von quotientensingularitäten (nach zyklischen gruppen). *Mathematische Annalen*, 209:211–248, 09 1974.

[Sch71] Michael Schlessinger. Rigidity of quotient singularities. *Inventiones mathematicae*, 14(1):17–26, Mar 1971.

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